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1974 J. Phys. A: Math. Nucl. Gen. 7 665

(http://iopscience.iop.org/0301-0015/7/6/003)

J. Phys. A: Math., Nucl. Gen., Vol. 7, No. 6, 1974. Printed in Great Britain. © 1974

Graphs and the quantization of the Gel'fand–Yaglom equations for higher spin

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Received 19 November 1973

Abstract. We give further details of a graphical approach to the Gel'fand–Yaglom equations and their quantization, in the case when repeated representations are not allowed. Graphs associated with the s blocks of L_0 are used in their construction, in writing down their characteristic polynomials and the traces required in the quantization conditions. As well as being useful as practical procedures, the methods described are more suggestive theoretically than the conventional algebraic treatment. As examples we obtain a range of quantizable spin-2 theories. In particular we show that it is possible to obtain various mass spectra and yet still have a quantizable theory. Such multi-mass theories have only been obtained previously using repeated representations. Also we prove that the representation $(-1, j+1) \oplus (0, j+1) \oplus (1, j+1)$ must occur in a good quantizable theory with maximum spin $j \neq 0$. Although we only consider integer spin here, the modifications for half-odd integer spin are straightforward.

1. Introduction

In a previous paper (Cox 1974) we have introduced a new graphical approach to the theory of the Gel'fand–Yaglom equations for higher spin. In our theories we assumed that repeated representations of the proper Lorentz group are not used, that the charge and energy density for physical states are non-zero and that the mass–spin states are non-degenerate. Some theoretical results were obtained and a systematic procedure for finding quantizable theories developed. In this paper we give further details of this approach, showing how the graphical representation can in practice aid the construction and investigation of the s blocks of the L_0 matrix, and giving some examples of quantizable theories to illustrate the methods described. For simplicity we confine ourselves to integer spin, although the modifications for half-odd integer spin are straightforward.

In § 2 we outline the algorithm for constructing the s blocks from their graphs, which are easy to write down. A general spin-2 theory is given as an example.

In § 3 we show how the graphs can be used in practice to write down the characteristic polynomials for the s blocks, again with an example. As an illustration of the type of general result suggested by the graphical representation we also prove that for any good theory of our general type, the corresponding graph must contain a particular pair of branches.

In § 4 we review the procedure for finding quantizable theories. We discuss the quantizable unique mass theories, which are particularly easy to deal with and have been studied by other authors (Amar and Dozzio 1972a, b, Capri and Shamaly 1971).

Section 5 contains the illustrative and straightforward spin 0, 1 theories.

In § 6 we study a wide range of spin-2 theories, some exhibiting mass spectra. Such multiple-mass Gel'fand–Yaglom theories have only been obtained previously using repeated representations (Amar and Dozzio 1972b). We also obtain the unique mass spin-2 theories first given by Capri and Shamaly (1971); these cannot be made to exhibit mass spectra.

In § 7 we briefly discuss the continuation to higher spin and explain our reason for believing that good theories of our type are improbable for spin greater than eight.

2. The construction of the s blocks

The notation we use is described in Cox (1974) and is basically that of Gel'fand *et al* (1963). We plot the irreducible representations τ_i of $\mathscr{L}_p$ as points in the (l_0, l_1) plane, where l_0, l_1 are both integers for integer spin theories. Those points corresponding to finite representations occupy a fan (the 'Bose fan') in the upper half plane, $l_1 > |l_0|$. Any finite-dimensional field theory without repeated representations will correspond to a finite subset of the points in this fan.

It is convenient to have a standard numbering system for all the representations in the Bose fan, and here we take this to be in ascending order from left to right as l_1 increases. Thus the number of the representation $\tau \equiv (l_0, l_1)$ will be $i = l_1^2 - l_1 + l_0 + 1$ and we denote the representation by τ_i.

We now briefly recap the Gel'fand–Yaglom theory for later reference. We consider a theory based on a representation $\mathscr{R}$ of $\mathscr{L}$ and the usual equation

$$(L_\mu \partial^\mu + \mathrm{i}\chi)\psi = 0. \tag{2.1}$$

In the canonical basis L_0 has the form

$$L_0 = [C^{\tau\tau'}_{sms'm'}]$$

where if $\tau \equiv (l_0, l_1)$ then s takes the spin values, $s = |l_0|, |l_0|+1, \ldots, l_1 - 1$ and m the values, $m = -s, -(s-1), \ldots, s-1, s$. Covariance under $\mathscr{L}_p$ leads to

$$C^{\tau\tau'}_{sms'm'} = C^{\tau\tau'}_s \, \delta_{ss'} \, \delta_{mm'}$$

where the $C^{\tau\tau'}_s$ are zero except for 'interlocked' or 'linked' representations for which $(l'_0, l'_1) = (l_0 \pm 1, l_1)$ or $(l'_0, l'_1) = (l_0, l_1 \pm 1)$. In this case the non-zero $C^{\tau\tau'}_s$ are given by: *Type* (*i*) *or horizontal linkage* $(l'_0, l'_1) = (l_0 + 1, l_1)$

$$\left.\begin{aligned} C^{\tau\tau'}_s &= \rho(s, l_0) C^{\tau\tau'} \\ C^{\tau'\tau}_s &= \rho(s, l_0) C^{\tau'\tau} \end{aligned}\right\} \tag{2.2}$$

Type (*ii*) *or vertical linkage* $(l'_0, l'_1) = (l_0, l_1 + 1)$

$$\left.\begin{aligned} C^{\tau\tau'}_s &= \rho(s, l_1) C^{\tau\tau'} \\ C^{\tau'\tau}_s &= \rho(s, l_1) C^{\tau'\tau} \end{aligned}\right\} \tag{2.3}$$

where the $C^{\tau\tau'}, C^{\tau'\tau}$ are arbitrary complex numbers, and where $\rho(s, n) = |\sqrt{(s+n+1)(s-n)}|$. We also demand that the theory be derivable from a lagrangian and that it be covariant to space reflections. Equation (2.1) is derivable from the lagrangian density

$$L[\psi(x)] = \frac{1}{\mathrm{i}} \psi^\dagger \Lambda (L_\mu \partial^\mu + \mathrm{i}\chi)\psi, \tag{2.4}$$

where the most general non-degenerate invariant hermitian form Λ in the representation space $\mathscr{R}$ is given by

$$\psi_2^\dagger \Lambda \psi_1 = \sum_{\tau s m} a^{\tau\tau^{\cdot}}(-1)^{[s]} x_{sm}^{\tau} \overline{y_{sm}^{\tau^{\cdot}}}, \tag{2.5}$$

where $\psi_1 = (x_{sm}^{\tau})$, $\psi_2 = (y_{sm}^{\tau})$, $[s]$ denotes the integer part of s and $\tau^{\cdot} = (-l_0, l_1)$ is the representation conjugate to τ. The $a^{\tau\tau^{\cdot}}$ satisfy $a^{\tau\tau^{\cdot}} = \overline{a^{\tau^{\cdot}\tau}}$ and in fact, by a suitable choice of canonical basis we can always ensure $a^{\tau\tau^{\cdot}} = \pm 1$, which we will assume in future. $L[\psi(x)]$ is invariant, and we only have to demand that it be real. This is so if and only if

$$L_0^\dagger \Lambda = \Lambda L_0 \tag{2.6}$$

and this imposes on the C parameters the extra condition

$$C^{\tau\tau'} = s(\tau, \tau')\overline{C^{\tau'^{\cdot}\tau^{\cdot}}}, \tag{2.7}$$

where

$$s(\tau, \tau') = \frac{a^{\tau\tau^{\cdot}}}{a^{\tau'\tau'^{\cdot}}} = \frac{a^{\tau'\tau'^{\cdot}}}{a^{\tau\tau^{\cdot}}} = \pm 1.$$

The further conditions imposed by space reflection covariance depend on the nature of the representations τ, τ'. There are three distinct cases:

(*a*) $\tau \neq \tau^{\cdot}$ *and* $\tau' \neq \tau'^{\cdot}$

$$C^{\tau\tau'} = C^{\tau^{\cdot}\tau'^{\cdot}}$$

which combined with (2.7) gives

$$C^{\tau'\tau} = s(\tau, \tau')\overline{C^{\tau'\tau}}.$$

(*b*) $\tau = \tau^{\cdot}$ *and* $\tau' \neq \tau'^{\cdot}$

Because of the self-conjugate representation τ, there are two inequivalent representations of the reflection operator S in this case, and each will lead to different conditions on the $C^{\tau\tau'}$ and possibly to different theories. Effectively the two possibilities can be summarized by the condition

$$C^{\tau\tau'} = \eta(\tau)C^{\tau^{\cdot}\tau'^{\cdot}}$$

where $\eta(\tau) = \pm 1$ depending on which representation of S is chosen. Combining this case with (2.7) gives

$$C^{\tau\tau'} = \eta(\tau)s(\tau, \tau')\overline{C^{\tau'\tau}}.$$

(*c*) $\tau = \tau^{\cdot}$ *and* $\tau' = \tau'^{\cdot}$

In this case no further conditions are imposed on the $C^{\tau\tau'}$, but the different possibilities for the reflection operator S can lead to different forms of the theories (eg tensor–psuedo tensor forms). In this case (2.7) becomes

$$C^{\tau\tau'} = s(\tau, \tau')\overline{C^{\tau'\tau}}.$$

(*a*), (*b*), (*c*) above give the final conditions on the C parameters such that (2.1) is covariant under $\mathscr{L}$ and is derivable from a lagrangian (2.4). We now divide L_0 into its s blocks

$$A_s = [C_s^{\tau\tau'}]$$

whose 'elements' are in fact scalar matrices. The representations τ which contribute non-zero elements to a particular s block will be those corresponding to points in or on the rectangle

$$l_0 = -s, \qquad l_0 = s$$

$$l_1 = s+1, \qquad l_1 = j+1$$

in the Bose fan, where j is the maximum value of s in the representation $\mathscr{R}$. As described in Cox (1974) we can construct a linear graph on this subset of points as nodes by inserting a directed branch from τ_i to τ_j if and only if the element $C_s^{\tau_i\tau_j}$ is non-zero. In this way we will obtain a subgraph of a lattice type graph which will represent pictorially the distribution of the non-zero elements of the s block. Each directed branch of the graph can be labelled with the corresponding element of A_s. It is not difficult to see what these elements will be from the above summary of the Gel'fand–Yaglom results. In practice, the advantage is actually in reversing the procedure to construct the s blocks for any particular theory by reference to the graphs.

First identify those representations in the 's rectangle', and write these in their standard order as the row and column indicators for the s block. The s block will then appear in triple block diagonal form, according to the l_1 rows of the s block graph. That is, the triple diagonal structure reflects the graph structure of rows linked to rows above and below. We call these sub-blocks of the s blocks 'ρ blocks'.

The diagonal ρ blocks correspond to the horizontal linkages along rows, so they have zeros everywhere except immediately above and below their diagonal. Bearing in mind that horizontal linkages are type (i), and collecting all the conditions on the $C^{\tau\tau'}$ listed earlier, the elements of a particular ρ block can be obtained from the typical diagram of a row in figure 1. In this figure $s_{ij} = s(\tau_i, \tau_j)$, $\eta_i = \eta(\tau_i)$, and $C_{ij} = C^{\tau_i\tau_j}$. Also, note that because $\rho(s, -r) = \rho(s, r-1)$, the $\rho(s, l_0)$ factors are symmetrical about the l_1 axis.

The off-diagonal ρ blocks correspond to vertical linkages between rows, and so only the diagonal elements can be non-zero. The ρ blocks above/below the diagonal of the s blocks correspond to branches directed upwards/downwards. Summarizing the

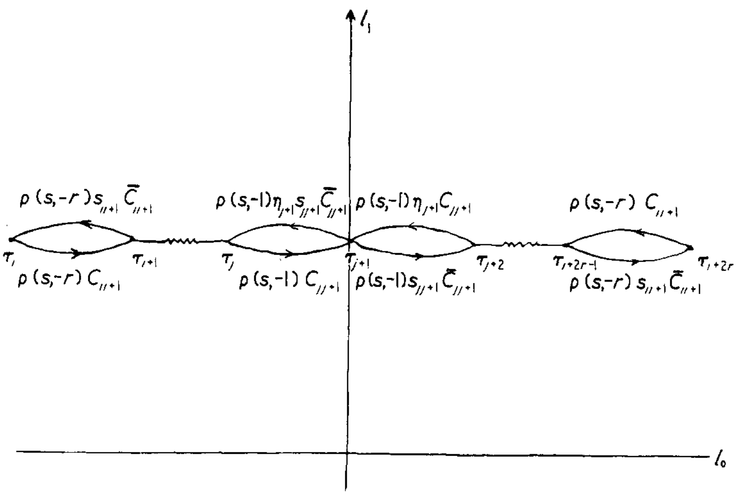

Figure 1. The graphs of typical type (i) (horizontal) linkages, the branches corresponding to the elements indicated.

conditions on the $C^{\tau\tau'}$ parameters, the off-diagonal ρ blocks can be constructed from the typical picture shown in figure 2. Note that in each of the off-diagonal ρ blocks, the $\rho(s, l_1)$ factor can be taken out as a common factor.

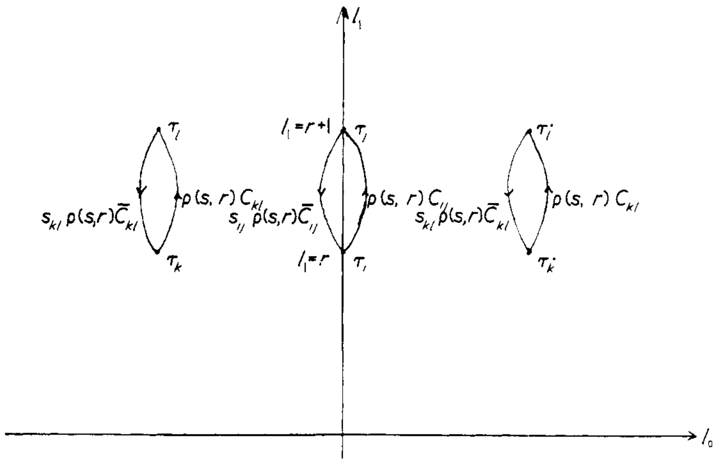

Figure 2. The graphs of typical type (ii) (vertical) linkages, the branches corresponding to elements indicated.

Further details of the construction of the s blocks from the graph can be found in Cox (1972). It is really nothing more than a diagrammatic representation of the Gel'fand–Yaglom results. In figures 1 and 2 note the obvious relation between elements corresponding to a branch and its (a) mirror image in the l_1 axis—this is space reflection covariance, (b) opposite—this is a real lagrangian origin plus space reflection.

We now consider an example, based on a general representation $\mathscr{R}$ of $\mathscr{L}$ in which no s value higher than two can occur, see figure 3. This should be sufficient to explain the

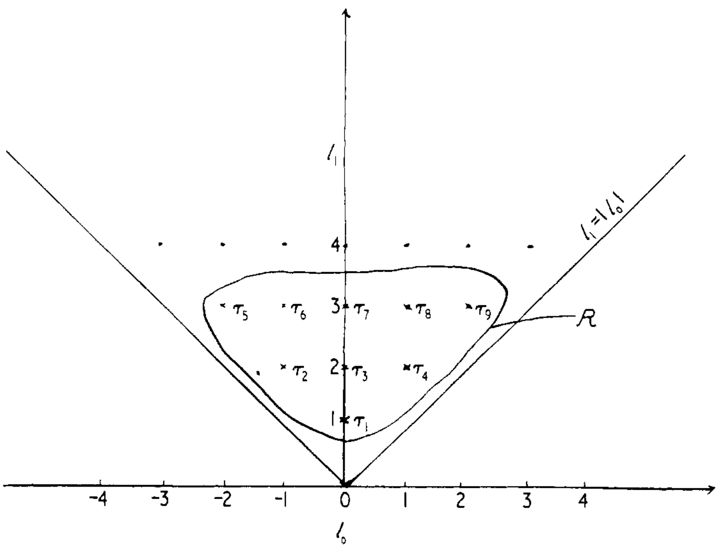

Figure 3. The subset of points corresponding to the representation $\mathscr{R}$ for a general maximum spin-2 theory.

method. The 0 block, 1 block, 2 block graphs are shown in figure 4 and respectively furnish us with the s block matrices:

0 block

$$\begin{array}{c c} & \begin{array}{ccc} 1 & 3 & 7 \end{array} \\ \begin{array}{c} 1 \\ 3 \\ 7 \end{array} & \left[\begin{array}{ccc} 0 & \sqrt{2}C_{13} & 0 \\ \sqrt{2}s_{13}\bar{C}_{13} & 0 & \sqrt{6}C_{37} \\ 0 & \sqrt{6}s_{37}\bar{C}_{37} & 0 \end{array}\right] \end{array}.$$

1 block

$$\begin{array}{c c} & \begin{array}{cccccc} 2 & 3 & 4 & 6 & 7 & 8 \end{array} \\ \begin{array}{c} 2 \\ 3 \\ 4 \\ 6 \\ 7 \\ 8 \end{array} & \left[\begin{array}{cccccc} 0 & \sqrt{2}C_{23} & 0 & 2C_{26} & 0 & 0 \\ \sqrt{2}\eta_3 s_{23}\bar{C}_{23} & 0 & \sqrt{2}s_{23}\bar{C}_{23} & 0 & 2C_{37} & 0 \\ 0 & \sqrt{2}\eta_3 C_{23} & 0 & 0 & 0 & 2C_{26} \\ 2s_{26}\bar{C}_{26} & 0 & 0 & 0 & \sqrt{2}C_{67} & 0 \\ 0 & 2s_{37}\bar{C}_{37} & 0 & \sqrt{2}\eta_7 s_{67}\bar{C}_{67} & 0 & \sqrt{2}s_{67}\bar{C}_{67} \\ 0 & 0 & 2s_{26}\bar{C}_{26} & 0 & \sqrt{2}\eta_7 C_{67} & 0 \end{array}\right] \end{array}.$$

2 block

$$\begin{array}{c c} & \begin{array}{ccccc} 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{array}{c} 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} & \left[\begin{array}{ccccc} 0 & 2C_{56} & 0 & 0 & 0 \\ 2s_{56}\bar{C}_{56} & 0 & \sqrt{6}C_{67} & 0 & 0 \\ 0 & \sqrt{6}\eta_7 s_{67}\bar{C}_{67} & 0 & \sqrt{6}s_{67}\bar{C}_{67} & 0 \\ 0 & 0 & \sqrt{6}\eta_7 C_{67} & 0 & 2s_{56}\bar{C}_{56} \\ 0 & 0 & 0 & 2C_{56} & 0 \end{array}\right] \end{array}.$$

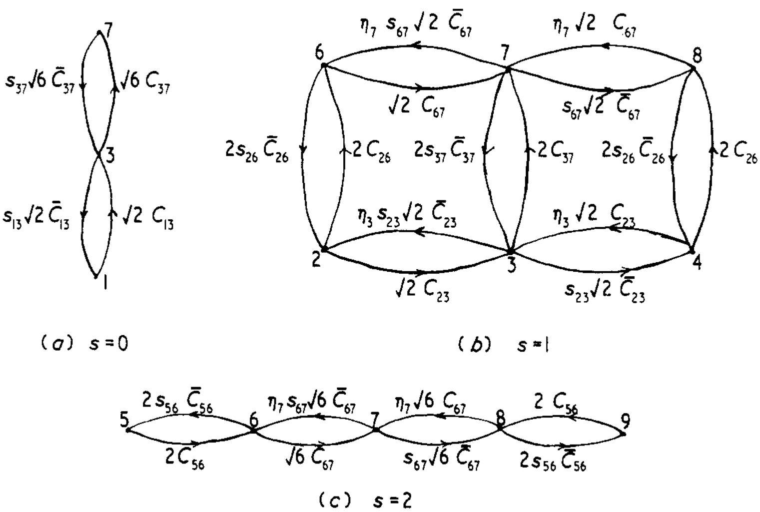

Figure 4. The s block graphs for the maximum spin-2 theory.

3. The characteristic polynomials of the s blocks

For our theories, with charge and energy density non-zero, and with non-degenerate mass–spin spectra, the characteristic and minimal polynomials of an $n \times n$ s block have the form

$$\Delta_s(t) = t^{n-2k} \prod_{i=1}^{k} (t^2 - m_i^2) \tag{3.1}$$

$$m(t) = t^q \prod_{i=1}^{k} (t^2 - m_i^2) \qquad n-2k \geqslant q \geqslant 1 \tag{3.2}$$

respectively where the m_i are all real and distinct (Cox 1974). Thus, an $n \times n$ s block A_s has characteristic polynomial

$$\Delta_s(t) \begin{cases} = tP_1(t^2) & \text{if } n \text{ odd} \\ = P_2(t^2) & \text{if } n \text{ even} \end{cases} \tag{3.3}$$

where P_1 and P_2 are polynomials. We now outline a graphical algorithm for constructing the $\Delta_s(t)$, which allows us to take advantage of the simple form of the s block graphs. The justification for the method is in the graph theoretical proof of (3.3) (Cox 1974). The method is, for small s blocks, quicker than direct determinantal expansion and we think is more suggestive from a theoretical point of view—it encourages us to use the graphs as a visual aid when investigating the s blocks.

Let G be any s block graph with n nodes and $A(G)$ the corresponding $n \times n$ s block matrix. The method for finding $\Delta(-t) = |A(G) - tI|$ consists of writing down all the contributions to the coefficient of $(-t)^r$ for each of the necessary values of r, taking into account the general form of $\Delta(-t)$ given in (3.3). Each possible combination of r nodes is inspected—these correspond to the terms containing $(-t)^r$ in the expansion of $|A(G) - tI|$—and the graph is searched for sets of disjoint loops not including these particular r nodes. If for some particular combination of r nodes, there is no such set of loops having a total of $n-r$ nodes, then the contribution of that particular combination of nodes to the term $(-t)^r$ is zero. Otherwise the factors corresponding to the sets of disjoint loops are written down as contributions to the coefficient of $(-t)^r$. This contribution will in fact be the product of all the s block elements corresponding to the branches contained in the set of disjoint loops, multiplied by $(-1)^l$ where l is the number of disjoint loops comprising the loop set.

To illustrate this procedure we will use the graph of the 1 block of the previous section (figure 4) to construct the $\Delta_1(t)$ for that spin-2 theory. We know from (3.3) that

$$\Delta_1(-t) = (-t)^6 + C_4(-t)^4 + C_2(-t)^2 + C_0$$

and we have to find the coefficients C_i by the above procedure.

C_2 and C_4. There are 15 possible pairs of nodes and 15 possible sets of four nodes. These two sets, complements of each other with respect to the set (2, 3, 4, 6, 7, 8), are conveniently tabulated together. Table 1 shows the node sets together with their contributions to the coefficients C_2, C_4, obtained by inspection of the graph.

Table 1.

C_4		C_2	
Nodes	Terms	Nodes	Terms
4678	$-(\sqrt{2}C_{23})(\eta_3 s_{23}\sqrt{2}\bar{C}_{23})$	23	$(\sqrt{2}C_{67})(\eta_7 s_{67}\sqrt{2}\bar{C}_{67})(2s_{26}\bar{C}_{26})(2C_{26})$
3678	0	24	0
3478	$-(2C_{26})(2s_{26}\bar{C}_{26})$	26	$(2C_{37})(2s_{37}\bar{C}_{37})(2C_{26})(2s_{26}\bar{C}_{26})$ $+(s_{23}\sqrt{2}\bar{C}_{23})(\eta_3\sqrt{2}C_{23})(s_{67}\sqrt{2}\bar{C}_{67})(\eta_7\sqrt{2}C_{67})$ $-(2s_{37}\bar{C}_{37})(s_{23}\sqrt{2}\bar{C}_{23})(2C_{26})(\eta_7\sqrt{2}C_{67})$ $-(2C_{37})(s_{67}\sqrt{2}\bar{C}_{67})(2s_{26}\bar{C}_{26})(\eta_3\sqrt{2}C_{23})$
3468	0	27	0
3467	0	28	$(\sqrt{2}C_{67})(\eta_7 s_{67}\sqrt{2}\bar{C}_{67})(\eta_3\sqrt{2}C_{23})(s_{23}\sqrt{2}\bar{C}_{23})$
2678	$-(s_{23}\sqrt{2}\bar{C}_{23})(\eta_3\sqrt{2}C_{23})$	34	$(2C_{26})(2s_{26}\bar{C}_{26})(s_{67}\sqrt{2}\bar{C}_{67})(\eta_7\sqrt{2}C_{67})$
2478	0		0
2468	$-(2C_{37})(2s_{37}\bar{C}_{37})$	37	$(2C_{26})(2s_{26}\bar{C}_{26})(2C_{26})(2s_{26}\bar{C}_{26})$
2467	0	38	0
2378	0	46	$(\sqrt{2}C_{23})(\eta_3 s_{23}\sqrt{2}\bar{C}_{23})(s_{67}\sqrt{2}\bar{C}_{67})(\eta_7\sqrt{2}C_{67})$
2368	0	47	0
2367	$-(2C_{26})(2s_{26}\bar{C}_{26})$	48	$(\sqrt{2}C_{23})(\eta_3 s_{23}\sqrt{2}\bar{C}_{23})(\sqrt{2}C_{67})(\eta_7 s_{67}\sqrt{2}\bar{C}_{67})$ $+(2C_{26})(2s_{26}\bar{C}_{26})(2s_{37}\bar{C}_{37})(2C_{37})$ $-(\sqrt{2}C_{67})(2s_{37}\bar{C}_{37})(\eta_3 s_{23}\sqrt{2}\bar{C}_{23})(2C_{26})$ $-(\eta_7 s_{67}\sqrt{2}\bar{C}_{67})(2s_{26}\bar{C}_{26})(\sqrt{2}C_{23})(2C_{37})$
2348	$-(\sqrt{2}C_{67})(\eta_7 s_{67}\sqrt{2}\bar{C}_{67})$	67	$(2C_{26})(2s_{26}\bar{C}_{26})(\sqrt{2}C_{23})(\eta_3 s_{23}\sqrt{2}\bar{C}_{23})$
2347	0	68	0
2346	$-(s_{67}\sqrt{2}\bar{C}_{67})(\eta_7\sqrt{2}C_{67})$	78	$(2C_{26})(2s_{26}\bar{C}_{26})(s_{23}\sqrt{2}\bar{C}_{23})(\eta_3\sqrt{2}C_{23})$

C_0. This is the sum of all terms corresponding to sets of disjoint loops containing a total of six branches. By inspection of the graph in figure 4 we find the following loop sets, with the signs indicated:

$$(26)(3784)+(26)(7348)+(48)(3762)+(48)(7326)-(26)(37)(48)-(48)(67)(23)$$
$$-(26)(78)(34)-(234876)-(326784).$$

Inserting the corresponding elements and simplifying gives C_0. We find

$$C_4 = -4(\eta_3 s_{23}p_{23}+2s_{26}p_{26}+s_{37}p_{37}+\eta_7 s_{67}p_{67})$$
$$C_2 = 16[\eta_7 s_{26}s_{67}p_{26}p_{67}+2s_{26}s_{37}p_{26}p_{37}+\eta_7\eta_3 s_{23}s_{67}p_{23}p_{67}+p_{26}^2+\eta_3 s_{23}s_{26}p_{23}p_{26} -(\eta_3+\eta_7)s_{23}s_{37}R]$$
$$C_0 = 32p_{26}[2(\eta_3+\eta_7)s_{67}R-2s_{37}p_{26}p_{37}-(\eta_3\eta_7+1)s_{37}p_{23}p_{67}],$$

where $p_{ij}=|C_{ij}|^2$ and $R=\mathrm{Re}(C_{23}C_{37}\bar{C}_{26}\bar{C}_{67})$. Also, we have used the fact that $s_{ik}s_{kj}=s_{ie}s_{ej}$.

Notice the simple general result that for an $n\times n$ s block the coefficient of t^{n-2} in $\Delta_s(t)$ is minus one times the sum of the terms corresponding to all the 2-loops of the s block graph.

We now prove a result concerning the graph of the j block where j is the maximum spin representation occurring in the theory ($j\neq 0$). Suppose that the j block graph, G, has the typical form shown in figure 5, ie the 'top-row' of the representation $\mathscr{R}$ has gaps

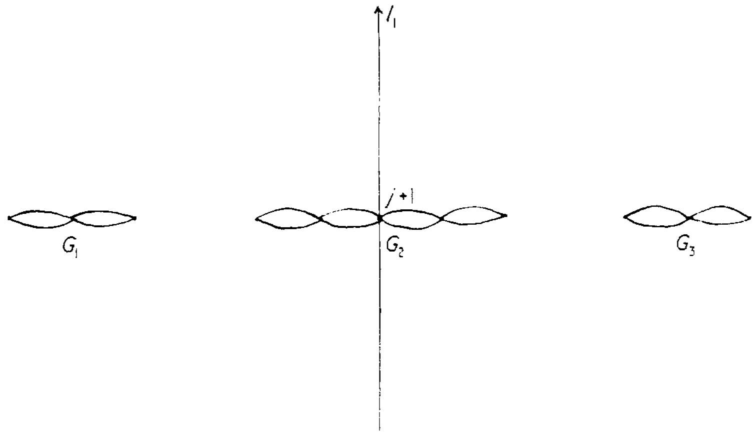

Figure 5. A disconnected j-block graph.

in it. The A block will then be reducible and take the form

$$A(G) = \begin{bmatrix} A(G_1) & 0 & 0 \\ 0 & A(G_2) & 0 \\ 0 & 0 & A(G_3) \end{bmatrix}$$

where, by the results of § 2:

$$A(G_3) = A^{\mathrm{T}}(G_1)$$

and so $A(G_1)$ and $A(G_3)$ have the same characteristic polynomials, $\Delta_1(t)$ say. If the characteristic polynomial of $A(G_2)$ is $\Delta_2(t)$ then that of $A(G)$ will be

$$\Delta(t) = \Delta_1^2(t)\Delta_2(t).$$

For our theories $\Delta(t)$ must have no repeated non-zero roots and so all of the roots of $\Delta_1(t)$ must be zero and the non-zero eigenvalues of $A(G)$ must come solely from $\Delta_2(t)$, ie from G_2. In particular, if G_2 consists of a single node, then $\Delta_2(t)$ has only zero roots and so $A(G)$ has all zero eigenvalues and there is no spin j state. This can only be avoided by ensuring that the j block graph is connected across the l_1 axis, although there may be gaps elsewhere in the j block. In terms of representations, a field theory with a state of maximum spin $j \neq 0$, non-degenerate mass–spin states and non-zero charge and energy densities must contain the linked representations $(-1, j+1), (0, j+1), (1, j+1)$.

The above result shows how we can restrict the forms of graph which can give good theories, and this limits the number of possibilities we have to consider when looking for such theories. Even if the above linkage is present this does not of course imply a good theory. In fact, by extending the above argument, we can eliminate an entire range of graphs—those which have a particular type of 'hole' in them. We will pursue this elsewhere. Having obtained the characteristic polynomial, $\Delta(t)$, we choose the η, s, C parameters to give it the exact form we require, using these parameters to fix the coefficients. In this way we impose the mass–spin spectra we require. It may happen that insufficient arbitrary parameters exist to do this, in which case the particular representation $\mathscr{R}$ cannot support that particular mass–spin spectrum.

4. The condition for a quantizable theory

We look for quantizable theories using the following procedure (Cox 1974). Let A be any s block of the theory with characteristic polynomial of the form (3.1). If possible we now choose the remaining arbitrary parameters such that A satisfies a polynomial equation of the form:

$$f(A) = A^p \prod_{i=1}^{k} (A^2 - m_i^2) = 0$$

where p is open to choice. $f(A)$ need not necessarily be the minimal polynomial of A. Different values of p may lead to different possible theories. The theory will be quantizable if and only if (subject to convention) for each such s block having non-zero eigenvalues:

if p odd

$$\operatorname{sgn}(T_r(p+1)) = (-1)^{k_r} \tag{4.1}$$

if p even

$$\operatorname{sgn}(T_r(p)) = (-1)^{k_r} \tag{4.2}$$

for each $r = 1, \ldots, k$ where

$$T_r(x) = \operatorname{Tr}\left(\Lambda_s A^x \prod_{i \neq r} (A^2 - m_i^2)\right) \tag{4.3}$$

and k_r is the number of $m_i > m_r$. In $T_r(x)$, Λ_s is of course the s block of Λ, ie the restriction of Λ to the s subspace. If there is just one pair of non-zero eigenvalues for a particular s block then k_r should be taken as zero to obtain the appropriate conditions. If no s block in the theory is allowed to have more than two non-zero eigenvalues, then the trace conditions take on the simpler form $\operatorname{Tr}(\Lambda_s A^l) > 0$ for each s block with non-zero eigenvalues. In this paper these are the only types of theory we consider, as the algebra for these is difficult enough, without the added complication of multiple masses with the same spin. The calculation of $\operatorname{Tr}(\Lambda_s A^l)$ can be facilitated by noting that the coefficient of $(\Lambda_s)_{\tau\tau'}$ in this trace is just the sum of the terms corresponding to all paths of length l from the representation τ to the representation τ. For small s blocks and small l and with some practice this provides a very quick way of writing out $\operatorname{Tr}(\Lambda_s A^l)$, and again is theoretically suggestive (Cox 1974).

One type of theory which is easily dealt with and has received much attention in the past is that of unique mass. In this case exactly one s block A_j must have a single pair of non-zero eigenvalues, the remaining s blocks being nilpotent. Only one trace condition has to be satisfied and in fact this is always possible by convention, since Λ is arbitrary up to a real multiplying factor. So in this type of theory all we have to do is ensure that the necessary characteristic polynomials are satisfied for each s block and our theory will be automatically quantizable. This is the basis of Capri's method. Capri and Shamaly (1971) start with the spinor representation of L_0 originally obtained by Bhabha (1945) and use Wild's transformation (Wild 1947) to obtain what is effectively the canonical representation of Gel'fand and Yaglom. They then make all s blocks except one nilpotent and use the Umezawa–Visconti relation (Umezawa and Visconti 1956) on this s block to ensure unique mass. In actual fact it is unnecessary to insist on the Umezawa–Visconti relation, which has recently been proved incorrect (Glass 1971).

Provided the required s block satisfies a polynomial which divides its characteristic polynomial we will get a good quantizable theory—with non-zero charge and energy density.

A particular case of the unique mass theory, discussed by Amar and Dozzio (1972a, b) is that where the non-nilpotent s block is required to be diagonalizable, ie its minimal polynomial is

$$m(t) = t(t^2 - m^2) \tag{4.4}$$

the trace condition then becomes $\mathrm{Tr}(\Lambda A^2) > 0$ and this, with (4.4) implies $\psi^\dagger \Lambda A^2 \psi > 0$ for our theories, where ψ is arbitrary. It is this condition which Amar and Dozzio use to obtain the restricted form of the graphs from which such unique mass theories can be drawn.

5. Spin-0 and spin-1 theories

We only consider the simplest of spin-0 and spin-1 theories, using the representation $(0, 1) \oplus (-1, 2) \oplus (0, 2) \oplus (1, 2)$ depicted in figure 6. Many more complicated theories are possible, for example those of Capri and Shamaly (1971), which may easily be checked. All we want here is a simple illustration of our approach. From figure 6 we find

0 block

$$A_0 = \begin{bmatrix} 0 & \sqrt{2}C_{13} \\ \sqrt{2}s_{13}\bar{C}_{13} & 0 \end{bmatrix}$$

with $\Delta_0(t) = t^2 - 2s_{13}p_{13}$, giving $s_{13} = +1$ for real mass, $\chi/\sqrt{2}|C_{13}|$. We now apply the results of § 4, noting that there is only one non-trivial choice for the minimal polynomial of the s block, and so the spin-0 state will be quantizable if

$$\mathrm{Tr}(\Lambda_0) > 0.$$

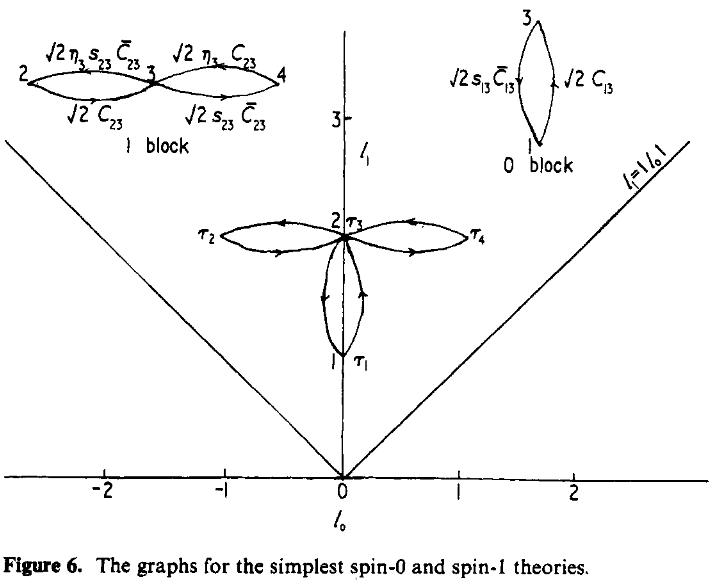

Figure 6. The graphs for the simplest spin-0 and spin-1 theories.

Using the matrix representation of Λ from (2.5), and $s_{13} = +1$, this is satisfied only if

$$a^{\tau_1\tau_1} = +1. \tag{5.1}$$

1 block

$$A_1 = \begin{bmatrix} 0 & \sqrt{2}C_{23} & 0 \\ \sqrt{2}\eta_3 s_{23}\bar{C}_{23} & 0 & \sqrt{2}s_{23}\bar{C}_{23} \\ 0 & \sqrt{2}\eta_3 C_{23} & 0 \end{bmatrix}$$

with $\Delta_1(t) = t(t^2 - 4\eta_3 s_{23} p_{23})$, giving $\eta_3 = s_{23}$ for real mass $\chi/2|C_{23}|$. Again there is only one non-trivial choice for the minimal polynomial and the spin-1 state will be quantizable if

$$\mathrm{Tr}(\Lambda_1 A_1^2) > 0.$$

We find,

$$\mathrm{Tr}(\Lambda_1 A_1^2) = (-1)^1[a^{\tau_2\tau_2}(2s_{23}p_{23}) + a^{\tau_3\tau_3}(4\eta_3 s_{23}p_{23}) + a^{\tau_2\tau_2}(2s_{23}p_{23})] = -8a^{\tau_3\tau_3}p_{23},$$

on using $\eta_3 s_{23} = +1$, from which

$$a^{\tau_3\tau_3} = -1. \tag{5.2}$$

(5.1) and (5.2) show that this representation cannot give a quantizable theory carrying both a spin-0 and spin-1 state, because they are incompatible with $s_{13} = +1$, the spin-0 real mass condition.

If we settle for just a spin-0 theory, then we must take $p_{23} = 0$, which reduces us to the theory based on the representation $(0, 1) \oplus (0, 2)$, which is the usual Duffin–Kemmer spin-0 theory. On the other hand, if we want a spin-1 theory, then we must take $p_{13} = 0$ and we get a theory based on the representation $(-1, 2) \oplus (0, 2) \oplus (1, 2)$, which is the usual spin-1 Duffin–Kemmer theory.

6. Spin-2 theories

We consider the general representation of figure 3, and initially assume that none of the C_{ij} are zero. The s blocks are given in § 2, and we study each in turn.

6.1. 0 block

$$\Delta_0(t) = t[t^2 - 2(s_{13}p_{13} + 3s_{37}p_{37})].$$

(*a*) A spin-0 state exists. In this case we must have

$$s_{13}p_{13} + 3s_{37}p_{37} > 0. \tag{6.1}$$

The minimal polynomial can only be $m(t) = \Delta_0(t)$ and from § 4 the spin-0 state will be quantizable if

$$\mathrm{Tr}(\Lambda_0 A_0^2) > 0$$

we find

$$\begin{aligned}\mathrm{Tr}(\Lambda_0 A_0^2) &= 2(-1)^0[a^{\tau_1\tau_1}(s_{13}p_{13}) + a^{\tau_3\tau_3}(s_{13}p_{13} + 3s_{37}p_{37}) + a^{\tau_7\tau_7}(3s_{37}p_{37})] \\ &= 2a^{\tau_3\tau_3}[(1+s_{13})p_{13} + 3(1+s_{37})p_{37}].\end{aligned}$$

From (6.1) this is positive if

$$a^{\tau_3\tau_3} = +1. \tag{6.2}$$

(*b*) Spin-0 state absent. In this case

$$s_{13}p_{13}+3s_{37}p_{37} = 0$$

whence

$$s_{13} = -s_{37} \quad \text{and} \quad p_{13} = 3p_{37}. \tag{6.3}$$

6.2. 1 block

The characteristic polynomial $\Delta_1(t)$ was given in § 3. With our simplicity requirement of no more than two non-zero eigenvalues we have:

(*a*) spin-1 state present:

$$\eta_3 r_{23}+2r_{26}+r_{37}+\eta_7 r_{67} > 0 \tag{6.4}$$

$$\eta_7 r_{26} r_{67}+2r_{26}r_{37}+\eta_3\eta_7 r_{23}r_{67}+r_{26}^2+\eta_3 r_{23}r_{26}-(\eta_3+\eta_7)s_{23}s_{37}R = 0 \tag{6.5}$$

$$r_{26}[2(\eta_3+\eta_7)s_{37}R-2r_{26}r_{37}-(\eta_3\eta_7+1)r_{23}r_{67}] = 0, \tag{6.6}$$

where $r_{ij} = s_{ij}p_{ij}$ and we have used $s_{ij}s_{kl} = s_{ik}s_{jl}$, etc. Note that in the above equation $\eta_3 = -\eta_7$ implies $p_{26} = 0$ or $p_{37} = 0$, which possibilities we exclude at the moment. So we have to take $\eta_3 = \eta_7$ in the above equation, which simplifies them a little. Thus the characteristic polynomial of the 1 block will be

$$\Delta_1(t) = t^4(t^2-m_1^2). \tag{6.7}$$

The minimal polynomial must then be one of

$$m(t) = t^r(t^2-m_1^2),$$

where $1 \leqslant r \leqslant 4$. Applying the results of § 4, we now have two ways of ensuring a quantizable spin-1 state:

(i) ensure that A_1 satisfies

$$m(t) = t^r(t^2-m_1^2), \tag{6.8}$$

where $r = 1$ or 2 and then take

$$\mathrm{Tr}(\Lambda_1 A_1^2) > 0. \tag{6.9}$$

Tedious algebra shows that (6.8) is satisfied (with $r = 2$) if

$$p_{26} = p_{23} = p_{67} = p_{37} \tag{6.10}$$

$$s_{23} = s_{67} \tag{6.11}$$

$$\eta_3 s_{23}s_{26} = -1 \tag{6.12}$$

$$C_{26}\bar{C}_{23}+C_{37}\bar{C}_{67} = C_{26}C_{67}+C_{23}C_{37} = 0 \tag{6.13}$$

(6.5) and (6.6) are then satisfied while (6.4) leads to

$$s_{37} = +1; \tag{6.14}$$

we find from the graph, and using (6.10)–(6.14) to simplify the result, that

$$\mathrm{Tr}(\Lambda_1 A_1^2) = -8a^{\tau_7\tau_7}p_{37}$$

so (6.9) gives

$$a^{\tau_7\tau_7} = -1 \tag{6.15}$$

which with (6.14) implies

$$a^{\tau_3\tau_3} = -1. \tag{6.16}$$

So, if we choose (6.10)–(6.16) then we get a good quantizable spin-1 state for the theory. Note that since (6.16) conflicts with (6.2), this theory could not contain both a spin-0 and a spin-1 state.

(ii) Allow complete freedom to the minimal polynomial, all we need to demand is the characteristic polynomial, and then take

$$\mathrm{Tr}(\Lambda_1 A_1^4) > 0. \tag{6.17}$$

So in this case we only need to satisfy (6.4)–(6.6), and then choose remaining arbitrary parameters such that (6.17) is satisfied. Since the left-hand side of (6.17) is a real expression in the η, s, C, we have altogether four real conditions involving four arbitrary complex numbers and six arbitrary signs, which can easily be satisfied irrespective of the value of $a^{\tau_3\tau_3}$. So with this form of the theory we could quite well have a spin-0 and spin-1 state together.

(*b*) Spin-1 state absent. In this case (6.5), (6.6) must be satisfied and also

$$\eta_3 r_{23} + 2r_{26} + r_{37} + \eta_7 r_{67} = 0. \tag{6.18}$$

6.3. 2 block

$$\Delta_2(t) = t[t^4 - 4(2s_{56}p_{56} + 3\eta_7 s_{67}p_{67})t^2 + 16p_{56}(p_{56} + 3\eta_7 s_{56}s_{67}p_{67})].$$

As we are interested only in spin-2 theories, we only consider the case where a spin-2 state is present, so

$$2s_{56}p_{56} + 3\eta_7 s_{67}p_{67} > 0$$

$$p_{56} + 3\eta_7 s_{56}s_{67}p_{67} = 0.$$

These give

$$s_{56} = -\eta_7 s_{67} = +1 \tag{6.19}$$

$$p_{56} = 3p_{67}. \tag{6.20}$$

The possible minimal polynomials for the 2 block are

$$m(t) = t^r(t^2 - m_2^2)$$

where $r = 1, 2, 3$, but the cases $r = 1, 2$ imply $p_{56} = p_{67} = 0$ so we only have to consider the case

$$m(t) = \Delta_2(t)$$

and so only have to demand

$$\mathrm{Tr}(\Lambda_2 A_2^4) > 0 \tag{6.21}$$

along with (6.19) and (6.20). We find, using these that $\mathrm{Tr}(\Lambda_2 A_2^4) = 48a^{\tau_7\tau_7}p_{56}p_{67}$ which results in

$$a^{\tau_7\tau_7} = +1 \tag{6.22}$$

from (6.21). This is compatible with a quantizable spin-0 state, but not with a spin-1 state of type (i) considered above. However with case (ii) of the 1 block we can have a quantizable theory carrying a spin-0, spin-1 and spin-2 state. In this theory, all representations indicated in figure 3 are used and none of the C_{ij} are zero. This may not be necessary of course, and it is easy to find alternative theories to these.

For example, omitting the representation τ_2 and τ_4 leaves the 0-block and 2-block theories unchanged, while the 1-block theory is considerably simplified. The new characteristic polynomial for the 1 block will be

$$\Delta_1(t) = t^2[t^2 - 4(s_{37}p_{37} + \eta_7 s_{67}p_{67})]$$

and for real mass we must have

$$s_{37}p_{37} + \eta_7 s_{67}p_{67} > 0. \tag{6.23}$$

This state will be quantizable if and only if

$$\mathrm{Tr}(\Lambda_1 A_1^2) > 0. \tag{6.24}$$

Using the new 1-block graph we find

$$\mathrm{Tr}(\Lambda_1 A_1^2) = -4a^{\tau_7\tau_7}[(s_{37}+1)p_{37} + (\eta_7 s_{67}+1)p_{67}]$$

from which (6.23) and (6.24) give

$$a^{\tau_7\tau_7} = -1.$$

If the spin-1 state is to be absent from the theory then

$$s_{37}p_{37} + \eta_7 s_{67}p_{67} = 0.$$

The above results, combined with those given for the 0, 2 blocks show that with this smaller representation we can satisfy all the conditions for a theory carrying:

(i) both a spin-0 and spin-2 state, but not a spin-1 state,

(ii) only a spin-2 state.

We can now also easily get the two spin-2 theories studied by Capri and Shamaly (1971). These are based on the representations

(iii) $\tau_2 \oplus \tau_3 \oplus \tau_4 \oplus \tau_6 \oplus \tau_7 \oplus \tau_8$

(iv) $\tau_1 \oplus \tau_3 \oplus \tau_6 \oplus \tau_7 \oplus \tau_8$

and have unique masses. We find that these representations cannot support multimass theories, because in both cases a quantizable spin-0 state is incompatible with a quantizable spin-2 state.

To summarize; while a complete study of spin-2 theories would be an arduous task, we have done sufficient to indicate our method and to show that with a big enough representation we can obtain quite general mass–spin spectra, without introducing repeated representations.

7. Higher spin theories—discussion

We have introduced a new approach to the Gel'fand–Yaglom equations, which simplifies and organizes to some extent the search for quantizable higher spin theories. To illustrate this we have obtained a number of spin-2 theories, some with multiple mass states. These examples also illustrate the great algebraic difficulties of finding higher spin theories. Because it is so difficult to test any particular theory for quantizability, it is important to look for general results which restrict the representations we have to consider, such as the result in § 3, or that of Amar and Dozzio (1972a, b). The graphical approach seems admirably suited to this because of the simple visual representations, which can be most suggestive. The general problem we have here is really one of using the simple structure of a graph to deduce statements about its associated matrix.

One important result is the conjecture that there are no quantizable theories of our type for spin greater than eight. For, assume that we have a general theory for a given integer maximum spin j, ie all of the representations in the Bose fan up to $l_1 = j+1$ are allowed (at most once), then by counting branches we find that there are $2j(j+1)$ arbitrary real contents in the theory, at most. We can work out the sizes of the s blocks, and using the known general form of the characteristic polynomials $\Delta_i(t)$ find the number of non-zero undetermined coefficients of these polynomials. For example, if j is even then the total number of coefficients (which are all real) in all the $\Delta_i(t)$ is $\frac{1}{12}j(j+2)(2j+5)$. For a particular theory we have to specify these coefficients, which means that this number of conditions has to be satisfied even before we demand quantizability. So roughly we could say that a quantizable theory is unlikely if

$$2j(j+1) < \frac{j(j+2)(2j+5)}{12}$$

ie if $j > 8$. A similar result is obtained by taking j odd. The basic idea is that as the spin increases, the number of conditions to be satisfied outstrips the number of available arbitrary parameters. It should therefore be possible in principle to catalogue all good theories not using repeated representations.

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